Solution to Assignment 9

Supplementary Exercise

1. Use the Weierstrass M-test to study the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $x \in (0, b)$ where $b > 0$.

   Solution. This series converges uniformly on any interval of the form $[0, b], b \in (0, 1)$ as a direct application of the M-Test. It is not convergent at $x = 1$, hence it cannot be uniformly convergent on $[0, b]$ when $b \geq 1$. We will see that this series has a closed form given by $-\log(1 - x)$.

2. Show that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ defines a continuous function on $\mathbb{R}$ for $p > 1$.

   Solution. By
$$\left| \frac{\sin nx}{n^p} \right| \leq \frac{1}{n^p}$$
   and $\sum n^{-p}$ is convergent if $p > 1$ we conclude from M-Test that this series is uniformly convergent on $\mathbb{R}$. As uniform convergence preserves continuity, the limit $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ is a continuous function.

3. Show that the infinite series $\sum_{j=1}^{\infty} \frac{\cos 2^j x}{3^j}$ is a continuous function on the real line. Is it differentiable?

   Solution. By
$$\left| \frac{\cos 2^j x}{3^j} \right| \leq \frac{1}{3^j}$$
   and $\sum_j 3^{-j} < \infty$, we conclude from M-Test that this series is uniformly convergent. By Continuity Theorem we further deduce that it is continuous. In fact, this function has continuous derivative given by
$$-\sum_{j=1}^{\infty} \left(\frac{2}{3} \right)^j \sin 2^j x \ .$$
   To see this just observe that $\sum_j (2/3)^j < \infty$ and then appeal to Differentiation Theorem.

4. Show that the sequence $g_n(x) = \sum_{j=1}^{n} e^{-jx}$ defines a smooth function on $[1, , \infty)$. What will happen if $[1, \infty)$ is replaced by $[0, \infty)$?

   Solution. By termwise differentiating this series k times we get the series
$$\sum_{j=1}^{\infty} (-1)^k j^k e^{-jx} \ .$$
   To show that the series defines a smooth function on $[1, \infty)$ it suffices to show these derived series are uniformly convergent and then apply Differentiation Theorem repeatedly. We recall the inequality
$$e^x \geq \frac{x^n}{n!}$$

for all positive x and n. In particular, taking x to be jx we get

$$e^{-jx} \le n!(jx)^{-n} \le n!j^{-n} ,$$

for $x \in [1, \infty)$. We choose $n = k+2$ to get $j^k e^{-jx} \le (k+2)!j^{-2}$. Since $\sum_j j^{-2} < \infty$, this series converges uniformly on $[1, \infty)$.

When considering the series on $[0, \infty)$, we note that it is not convergent at $x = 0$, so not even pointwise convergence, let alone uniform convergence.

Note: By slightly modifying the proof, this series is uniformly convergent on $[b, \infty)$ for every $b > 0$ and hence smooth on $(0, \infty)$.

5. (a) Suppose that $\sum_{n=1}^\infty f_n(x)$ is pointwisely convergent on E and g is a function on E. Show that $\sum_{n=1}^\infty g(x)f_n(x)$ pointwisely converges to $g(x)\sum_{n=1}^\infty f_n(x)$, that is,

$$\sum_{n=1}^\infty g(x)f_n(x) = g(x)\sum_{n=1}^\infty f_n(x) .$$

(b) Suppose further that $\sum_n f_n$ converges uniformly and g is bounded, show that $\sum_n gf_n$ converges uniformly.

Solution. (a) That $\sum_n f_n(x)$ is convergent means that for every $\varepsilon > 0$, there exists some n_0 (depending on x) such that

$$\left|\sum_{j=1}^n f_j(x) - \sum_{n=1}^\infty f_j(x)\right| < \frac{\varepsilon}{1+|g(x)|} , \quad \forall n \ge n_0 .$$

Therefore,

$$\left|\sum_{j=1}^n g(x)f_j(x) - g(x)\sum_{n=1}^\infty f_j(x)\right| < \frac{\varepsilon|g(x)|}{1+|g(x)|} < \varepsilon , \forall n \ge n_0 .$$

It shows that $\sum_n g(x)f_n(x)$ is convergent and

$$\sum_{j=1}^\infty g(x)f_n(x) = g(x)\sum_{j=1}^\infty f_j(x) .$$

(b) Let M satisfy $|g(x)| \le M$ for all $x \in E$. For $\varepsilon > 0$, there is some n_0 such that

$$\left\|\sum_{j=1}^n f_j - \sum_{j=1}^\infty f_j\right\| < \frac{\varepsilon}{M+1} , \forall n \ge n_0 .$$

Therefore,

$$\left\|\sum_{j=1}^n gf_j - g\sum_{j=1}^\infty f_j\right\| < \frac{\varepsilon M}{M+1} < \varepsilon , \quad \forall n \ge n_0 ,$$

from which the desired conclusion follows.

6. Suppose f is a nonzero function satisfying $f(x+y) = f(x)f(y)$ for all real numbers x and y and is differentiable at $x = 0$. Show that it must be of the form e^{ax} for some number a. Hint: Study the differential equation f satisfies.

Solution. Taking $x = y = 0$ in the defining relation of f we get $f(0) = f(0)^2$, so $f(0) = 0$ or $f(0) = 1$. We first exclude the case $f(0) = 0$. Indeed, from the relation $f(x + y) = f(x)f(y)$ one deduces $f(x) = f(x/n)^n$. If $f(0) = 0$, as $f'(0)$ exists, one has $\lim_{h\to 0} f(h)/h = 0$. For $\varepsilon = 1$, there exists some δ $|f(h)| \le |h|$ for all $|h| < \delta$. For any x we can find a large n_0 such that $|x|/n_0 < \delta$. Then $|f(x)| = |f(x/n)|^n \le (x/n)^n$ for all $n \ge n_0$. Letting $n \to \infty$, we conclude $f(x) = 0$ for all x, contradicting the assumption that f is non-zero. We have shown that $f(0) = 1$. Now,

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x)(f(h) - 1)}{h} \to f(x)f'(0) , \quad \text{as } h \to 0 ,$$

which shows that f is differentiable everywhere with $f'(x) = f'(0)f(x)$. Setting $a = f'(0)$, the function $g(x) = f(x/a)$ satisfies $g'(x) = a^{-1}f'(x/a) = a^{-1}f'(0)f(x/a) = g(x)$ and $g(0) = f(0) = 1$. By the uniqueness of the exponential function we conclude that $g(x) = E(x)$, so $f(x) = E(ax)$.

7. (a) Show that

$$1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} \le E(x) \le 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^a x^n}{n!} , \quad x \in [0, a] .$$

(b) Show that e is not a rational number. Suggestion: Deduce from (a) the inequality

$$0 < en! - \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) n! < \frac{e}{n+1} .$$

Solution. (a) A direct application of Taylor's Expansion Theorem.

(b) It follows from (a) by noting $e^a \le e$ for $a \in (0, 1]$. If $e = p/q$ for some $p, q \in \mathbb{N}$, the inequality gives

$$0 < \frac{p}{q} n! - \left(1 + \frac{1}{2} + \cdots \frac{1}{n!}\right) n! < \frac{e}{n+1} < 1 .$$

The term

$$\left(1 + \frac{1}{2} + \cdots \frac{1}{n!}\right) n!$$

is a natural number. When $n = q$ the term $\frac{p}{q} n!$ is also a natural number, so is their difference. But there is no natural number lying between 0 and 1 ! BTW, it is much more difficult to show e is a transcendental number. Google for it.

8. Show that the series

$$\sum_{j=0}^{\infty} \frac{x^j}{j!}$$

is not uniformly convergent on $\mathbb{R}$ (although it is uniformly convergent in every $[-M, M]$).

Solution. Were it uniformly convergent on $(-\infty, \infty)$, for $\varepsilon > 0$, there is some n_0 such that

$$\left| \sum_{j=0}^{n} \frac{x^j}{j!} - \sum_{j=0}^{m} \frac{x^j}{j!} \right| < \varepsilon, \quad \forall n, m \ge n_0 , \forall x \in (-\infty, \infty).$$

Taking $n = n_0 + 1$ and $m = n_0$, we have

$$\left| \frac{x^n}{n!} \right| < \varepsilon , \quad x \in (-\infty, \infty) ,$$

which is absurd. Hence the convergence cannot be uniform.

9. Optional. Let a be a positive number and $n \in \mathbb{N}$.

   (a) Show that there is a unique positive number b satisfying $b^n = a$. Write $b = a^{1/n}$.

   (b) For any rational number $m/n, m \in \mathbb{Z}, n \in \mathbb{N}$, define $a^{m/n} = (a^m)^{1/n}$. Show that $a^{m/n} = (a^{1/n})^m$.

   (c) Show that $a^{r_1+r_2} = a^{r_1}a^{r_2}$ for rational numbers r_1, r_2.

   This is 2050 stuff. It serves to refresh your memory.